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THE LOGIC OF CONSTRUCTIVISM

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ABSTRACT

In this paper I dispute the current view that intuitionistic logic is the common basis for the three main trends of constructivism in the philosophy of mathematics: intuitionism, Russian constructivism and Bishop's constructivism. The point is that the so-called 'Markov's principle', which is accepted by Russian constructivists and rejected by the other two, is expressible in intuitionistic first-order logic, and so it appears to have the status of a logical principle. The result of appending this principle to a complete intuitionistic axiom system for first-order predicate logic constitutes a new logic, which could well be called 'Markov's logic', and which should be regarded as the true logical system underlying Russian constructivism.

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1. INTUITIONISTIC LOGIC AND MARKOV'S PRINCIPLE

The purpose of Heyting's axiomatization of intuitionistic predicate logic was no doubt the codification of intuitionistic mathematical reasoning only. However, today intuitionistic logic is widely considered as representative of the two other main trends of constructivism in the philosophy of mathematics: Russian constructivism and Bishop's constructivism. Thus, in general handbooks on constructivism, in which all these three schools are represented, it is assumed that intuitionistic logic is *the* logic of constructive mathematics as a whole, and the terms 'constructive logic' and 'intuitionistic logic' are used as equivalent:

> In discussing pure logic we shall treat 'constructive' and 'intuitionistic' as synonymous.
> (Troelstra and van Dalen, 1988, p. 9.)
>
> […] a logic for constructive mathematics — *first-order intuitionistic logic* — […].
> (Bridges and Richman, 1987, p. 11, their italics.)

Hence, according to this established view, intuitionistic predicate logic is an adequate logical system to serve as a basis for Russian constructivism and Bishop's constructivism as much as for intuitionistic mathematics.

On the other hand, the main source of disagreement between intuitionistic mathematicians and Russian constructivists, followers of Markov, is the so-called 'Markov's principle', according to which if P is a decidable property, then the *non-non-existence* of an object satisfying P entails the existence of one, that is: $\neg\neg\exists x\, Px$ entails $\exists x\, Px$. In other words: if we can prove the absurdity of the supposition $\neg\exists x\, Px$, we can freely assume $\exists x\, Px$. This principle is accepted by Russian constructivists, and rejected by intuitionistic mathematicians as well as by Bishop and his followers.

Given the intuitionistic meaning of the universal quantifier, decidability is easily expressible as a universally quantified excluded middle; for example,

$$\forall x\, (Px \vee \neg Px)$$

is intuitionistically assertible if and only if we posses a constructive procedure to determine whether Px is the case or not for every object x in the domain, i.e., if the property P is decidable.

Hence, Markov's principle can be perfectly encapsulated as a logical rule, within the intuitionistic interpretation of the quantifiers:

$$\forall x\, (Px \vee \neg Px) \vdash \neg\neg\, \exists x\, Px \rightarrow \exists x\, Px$$

or, more in general, for an arbitrary formula α,

$$\forall x_1 x_2 \ldots x_n (\alpha \vee \neg\alpha) \vdash \neg\neg\exists x_1 x_2 \ldots x_n\, \alpha \rightarrow \exists x_1 x_2 \ldots x_n\, \alpha$$

(cf. e.g. Troelstra and van Dalen, 1988, p. 203).

2. MARKOV'S LOGIC

The fact that Markov's principle can be adequately formulated in purely logical terms — that is to say, in intuitionistically logical terms — means that it has the status of a *logical* principle. It is certainly *not* a principle of intuitionistic logic: the decision procedure for a property P together with the proof of the absurdity of $\neg\exists x\, Px$ does *not* necessarily produce a particular construction which satisfies P. But Markov's principle is indeed a principle of Russian constructivism, and, having the status of a logical principle, should be embedded as an axiom — or theorem — in any logical system intended to serve as a basis for it.

Therefore we must consider that logical system which results from intuitionistic predicate logic after the addition of Markov's principle as an extra axiom. Given that Markov's principle is, within classical logic, trivially true, the resulting logic will be, so to speak, 'intermediate' between intuitionistic and classical logic. This new logic could well be called 'Markov's logic'.

Markov's logic requires an independent study from both the syntactical and the semantical point of view. In particular, the usual explanations of the intuitionistic meaning of the logical constants, which I have discussed in some detail elsewhere (cf. e.g. Fernández Díez 2000), will not be adequate to it, since these explanations render Markov's principle invalid.

3. THE LOGICAL STATUS OF MARKOV'S PRINCIPLE

It has sometimes been suggested that Markov's principle could be a non-logical, purely mathematical principle:

> […] MP [Markov's principle] is no principle of intuitionistic *logic*. What remains highly contestable is the implicit claim […] that, if MP be true at all, it must be true as a matter of logic (given, perhaps, some semantical reflections).
>
> One might agree […] over the status of MP in logic (plus, perhaps, semantics) and continue to maintain that MP is a mathematically correct statement governing the behaviour of the constructions which guarantee the intuitionistic truth of statements such as '*Pn* is decidable'. (D. C. McCarty, 1994, p. 105, his italics.)

McCarty appears to maintain that Markov's principle is a mathematical, and not a logical principle, because it applies exclusively to decidable relations. Hence, according to this line of argument, it will not cover *all* relations, but only a mathematical fraction, and for this reason it will not qualify as a logical — absolutely general — principle.

However, the point is that, within an intuitionistic language, decidability is a *logical* attribute and it is expressible in purely logical terms. Indeed, the principle as such applies to all relations: it is only the premise of it (the universally quantified excluded middle) which selects those relations which are decidable, in order to impose on them the condition expressed by the conclusion of the principle (the condition that *non-non-existence* implies existence). It is precisely the fact that this 'selection' can be made in solely logical terms, which makes the whole principle expressible in pure logic, and hence renders it a purely logical principle.

For the same reason, Markov's principle is *not* expressible in classical logic, for example, precisely because under a classical interpretation the decidability of the property corresponding to P does not reduce to the assertion $\forall x\,(Px \vee \neg Px)$.

And once it has been shown that the principle is expressible in a pure first-order logical vocabulary, it would seem very artificial to continue to regard it as a mathematical, and not a logical principle, and not to take it into account, e.g. for a semantical explanation of the use of the logical operators of Russian constructivism.

4. A COMPARISON WITH THE STATUS OF INTUITIONISTIC LOGIC IN PHYSICS

In connection with this point we should notice the following: constructive mathematics has been repeatedly criticized on the grounds that it is not powerful enough for serving the needs of our most successful physical theories (e.g. Putnam, 1975, p. 75). Nevertheless, most intuitionists will agree that, whether this is strictly true or not, it does not pose a problem for constructivism as such, which relates to pure, rather than applied mathematics.

In particular, an observant intuitionistic mathematician could probably agree on the use of classical principles as part of a physical theory, as long as it is employed to obtain results about the physical reality only:

> It is perfectly intelligible, even if in fact false, to say that there are infinitely many stars, or again, that a ball bounces infinitely often before coming to rest. The meaning of saying that some totality, of stars or of bounces, is infinite re-

> lates to the incompletability of the process of counting them: but the members of the totality are not generated by that process, and so the totality can be given to us by means of a concept which does not itself determine the size of the totality; there is therefore no absurdity in thinking of an infinite totality as already formed.
>
> From an intuitionistic standpoint, such a defence, however licit it may be when applied to empirically given objects or events, cannot be applied to mathematical totalities, whose elements are mental constructions. (Dummett, 2000, p. 42.)

Similarly, there is no reason in principle, for instance, why an observant intuitionist could not use the law of excluded middle when applied to real existing objects, even if he cannot determine which of the two opposite options holds; and in doing this he would be effectively treating the law of excluded middle as a physical (empirical) law, in spite of being expressible in pure logical terms.

The situation, however, is quite different from that of the use of Markov's principle in Russian recursive mathematics, since in that case the principle in question is held by Markov and his followers as valid in every domain, and not only in a restricted area of phenomena. And notice again that in Markov's principle the requirement that the relation in question be decidable is contained within the statement of the principle: hence the principle as such is universal. This is why a distinction between Markov's principle and the other logical principles that Markov's followers accept would be purely ad hoc.

On the contrary, the requirement that the law of excluded middle be applied only to real existing objects, is external to the law of excluded middle itself, and could never be contained within it, since the predicate of 'empirical existence' is not expressible in logical or mathematical terms.

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